Chapter 5
Dimensional Analysis and Similarity

Motivation. In this chapter we discuss the planning, presentation, and interpretation of experimental data. We shall try to convince you that such data are best presented in *dimensionless* form. Experiments which might result in tables of output, or even multiple volumes of tables, might be reduced to a single set of curves—or even a single curve—when suitably nondimensionalized. The technique for doing this is *dimensional analysis*.

Chapter 3 presented gross control-volume balances of mass, momentum, and energy which led to estimates of global parameters: mass flow, force, torque, total heat transfer. Chapter 4 presented infinitesimal balances which led to the basic partial differential equations of fluid flow and some particular solutions. These two chapters covered *analytical* techniques, which are limited to fairly simple geometries and well-defined boundary conditions. Probably one-third of fluid-flow problems can be attacked in this analytical or theoretical manner.

The other two-thirds of all fluid problems are too complex, both geometrically and physically, to be solved analytically. They must be tested by experiment. Their behavior is reported as experimental data. Such data are much more useful if they are expressed in compact, economic form. Graphs are especially useful, since tabulated data cannot be absorbed, nor can the trends and rates of change be observed, by most engineering eyes. These are the motivations for dimensional analysis. The technique is traditional in fluid mechanics and is useful in all engineering and physical sciences, with notable uses also seen in the biological and social sciences.

Dimensional analysis can also be useful in theories, as a compact way to present an analytical solution or output from a computer model. Here we concentrate on the presentation of experimental fluid-mechanics data.

5.1 Introduction

Basically, dimensional analysis is a method for reducing the number and complexity of experimental variables which affect a given physical phenomenon, by using a sort of compacting technique. If a phenomenon depends upon n dimensional variables, dimensional analysis will reduce the problem to only k *dimensionless* variables, where the reduction $n - k = 1, 2, 3$, or 4, depending upon the problem complexity. Generally $n - k$ equals the number of different dimensions (sometimes called basic or primary or fundamental dimensions) which govern the problem. In fluid mechanics, the four basic dimensions are usually taken to be mass M, length L, time T, and temperature Θ, or an $MLT\Theta$ system for short. Sometimes one uses an $FLT\Theta$ system, with force F replacing mass.

Although its purpose is to reduce variables and group them in dimensionless form, dimensional analysis has several side benefits. The first is enormous savings in time and money. Suppose one knew that the force F on a particular body immersed in a stream of fluid depended only on the body length L, stream velocity V, fluid density ρ, and fluid viscosity μ, that is,

$$F = f(L, V, \rho, \mu) \tag{5.1}$$

Suppose further that the geometry and flow conditions are so complicated that our integral theories (Chap. 3) and differential equations (Chap. 4) fail to yield the solution for the force. Then we must find the function $f(L, V, \rho, \mu)$ experimentally.

Generally speaking, it takes about 10 experimental points to define a curve. To find the effect of body length in Eq. (5.1), we have to run the experiment for 10 lengths L. For each L we need 10 values of V, 10 values of ρ, and 10 values of μ, making a grand total of 10^4, or 10,000, experiments. At \$50 per experiment—well, you see what we are getting into. However, with dimensional analysis, we can immediately reduce Eq. (5.1) to the equivalent form

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \tag{5.2}$$

or

$$C_F = g(\mathrm{Re})$$

i.e., the dimensionless *force coefficient* $F/(\rho V^2 L^2)$ is a function only of the dimensionless *Reynolds number* $\rho VL/\mu$. We shall learn exactly how to make this reduction in Secs. 5.2 and 5.3.

The function g is different mathematically from the original function f, but it contains all the same information. Nothing is lost in a dimensional analysis. And think of the savings: We can establish g by running the experiment for only 10 values of the single variable called the Reynolds number. We do not have to vary L, V, ρ, or μ separately but only the *grouping* $\rho VL/\mu$. This we do merely by varying velocity V in, say, a wind tunnel or drop test or water channel, and there is no need to build 10 different bodies or find 100 different fluids with 10 densities and 10 viscosities. The cost is now about \$500, maybe less.

A second side benefit of dimensional analysis is that it helps our thinking and planning for an experiment or theory. It suggests dimensionless ways of writing equations before we waste money on computer time to find solutions. It suggests variables which can be discarded; sometimes dimensional analysis will immediately reject variables, and at other times it groups them off to the side, where a few simple tests will show them to be unimportant. Finally, dimensional analysis will often give a great deal of insight into the form of the physical relationship we are trying to study.

A third benefit is that dimensional analysis provides *scaling laws* which can convert data from a cheap, small *model* to design information for an expensive, large *prototype*. We do not build a million-dollar airplane and see whether it has enough lift force. We measure the lift on a small model and use a scaling law to predict the lift on the full-scale prototype airplane. There are rules we shall explain for finding scaling laws. When the scaling law is valid, we say that a condition of *similarity* exists between the model and the prototype. In the simple case of Eq. (5.1), similarity is achieved if the Reynolds number is the same for the model and prototype because the function g then requires the force coefficient to be the same also:

$$\text{If} \quad \mathrm{Re}_m = \mathrm{Re}_p \quad \text{then} \quad C_{Fm} = C_{Fp} \tag{5.3}$$

where subscripts m and p mean model and prototype, respectively. From the definition of force coefficient, this means that

$$\frac{F_p}{F_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 \tag{5.4}$$

for data taken where $\rho_p V_p L_p/\mu_p = \rho_m V_m L_m/\mu_m$. Equation (5.4) is a scaling law: If you measure the model force at the model Reynolds number, the prototype force at the same Reynolds number equals the model force times the density ratio times the velocity ratio squared times the length ratio squared. We shall give more examples later.

Do you understand these introductory explanations? Be careful; learning dimensional analysis is like learning to play tennis: There are levels of the game. We can establish some ground rules and do some fairly good work in this brief chapter, but dimensional analysis in the broad view has many subtleties and nuances which only time and practice and maturity enable you to master. Although dimensional analysis has a firm physical and mathematical foundation, considerable art and skill are needed to use it effectively.

EXAMPLE 5.1

A copepod is a water crustacean approximately 1 mm in diameter. We want to know the drag force on the copepod when it moves slowly in fresh water. A scale model 100 times larger is made and tested in glycerin at $V = 30$ cm/s. The measured drag on the model is 1.3 N. For similar conditions, what are the velocity and drag of the actual copepod in water? Assume that Eq. (5.1) applies and the temperature is 20°C.

Solution

From Table A.3 the fluid properties are:

Water (prototype): $\mu_p = 0.001$ kg/(m · s) $\quad \rho_p = 998$ kg/m^3

Glycerin (model): $\mu_m = 1.5$ kg/(m · s) $\quad \rho_m = 1263$ kg/m^3

The length scales are $L_m = 100$ mm and $L_p = 1$ mm. We are given enough model data to compute the Reynolds number and force coefficient

$$\mathrm{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \frac{(1263\ \text{kg/m}^3)(0.3\ \text{m/s})(0.1\ \text{m})}{1.5\ \text{kg/(m}\cdot\text{s)}} = 25.3$$

$$C_{Fm} = \frac{F_m}{\rho_m V_m^{\,2} L_m^{\,2}} = \frac{1.3\ \text{N}}{(1263\ \text{kg/m}^3)(0.3\ \text{m/s})^2(0.1\ \text{m})^2} = 1.14$$

Both these numbers are dimensionless, as you can check. For conditions of similarity, the prototype Reynolds number must be the same, and Eq. (5.2) then requires the prototype force coefficient to be the same

$$\text{Re}_p = \text{Re}_m = 25.3 = \frac{998V_p(0.001)}{0.001}$$

or $$V_p = 0.0253 \text{ m/s} = 2.53 \text{ cm/s} \qquad \textit{Ans.}$$

$$C_{Fp} = C_{Fm} = 1.14 = \frac{F_p}{998(0.0253)^2(0.001)^2}$$

or $$F_p = 7.31 \times 10^{-7} \text{ N} \qquad \textit{Ans.}$$

It would obviously be difficult to measure such a tiny drag force.

Historically, the first person to write extensively about units and dimensional reasoning in physical relations was Euler in 1765. Euler's ideas were far ahead of his time, as were those of Joseph Fourier, whose 1822 book *Analytical Theory of Heat* outlined what is now called the *principle of dimensional homogeneity* and even developed some similarity rules for heat flow. There were no further significant advances until Lord Rayleigh's book in 1877, *Theory of Sound,* which proposed a "method of dimensions" and gave several examples of dimensional analysis. The final breakthrough which established the method as we know it today is generally credited to E. Buckingham in 1914 [29], whose paper outlined what is now called the *Buckingham pi theorem* for describing dimensionless parameters (see Sec. 5.3). However, it is now known that a Frenchman, A. Vaschy, in 1892 and a Russian, D. Riabouchinsky, in 1911 had independently published papers reporting results equivalent to the pi theorem. Following Buckingham's paper, P. W. Bridgman published a classic book in 1922 [1], outlining the general theory of dimensional analysis. The subject continues to be controversial because there is so much art and subtlety in using dimensional analysis. Thus, since Bridgman there have been at least 24 books published on the subject [2 to 25]. There will probably be more, but seeing the whole list might make some fledgling authors think twice. Nor is dimensional analysis limited to fluid mechanics or even engineering. Specialized books have been written on the application of dimensional analysis to metrology [26], astrophysics [27], economics [28], building scale models [36], chemical processing pilot plants [37], social sciences [38], biomedical sciences [39], pharmacy [40], fractal geometry [41], and even the growth of plants [42].

5.2 The Principle of Dimensional Homogeneity

In making the remarkable jump from the five-variable Eq. (5.1) to the two-variable Eq. (5.2), we were exploiting a rule which is almost a self-evident axiom in physics. This rule, the *principle of dimensional homogeneity* (PDH), can be stated as follows:

> If an equation truly expresses a proper relationship between variables in a physical process, it will be *dimensionally homogeneous;* i.e., each of its additive terms will have the same dimensions.

All the equations which are derived from the theory of mechanics are of this form. For example, consider the relation which expresses the displacement of a falling body

$$S = S_0 + V_0 t + \tfrac{1}{2}gt^2 \tag{5.5}$$

Each term in this equation is a displacement, or length, and has dimensions $\{L\}$. The equation is dimensionally homogeneous. Note also that any consistent set of units can be used to calculate a result.

Consider Bernoulli's equation for incompressible flow

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{const} \tag{5.6}$$

Each term, including the constant, has dimensions of velocity squared, or $\{L^2T^{-2}\}$. The equation is dimensionally homogeneous and gives proper results for any consistent set of units.

Students count on dimensional homogeneity and use it to check themselves when they cannot quite remember an equation during an exam. For example, which is it:

$$S = \tfrac{1}{2}gt^2? \qquad \text{or} \qquad S = \tfrac{1}{2}g^2t? \tag{5.7}$$

By checking the dimensions, we reject the second form and back up our faulty memory. We are exploiting the principle of dimensional homogeneity, and this chapter simply exploits it further.

Equations (5.5) and (5.6) also illustrate some other factors that often enter into a dimensional analysis:

Dimensional variables are the quantities which actually vary during a given case and would be plotted against each other to show the data. In Eq. (5.5), they are S and t; in Eq. (5.6) they are p, V, and z. All have dimensions, and all can be nondimensionalized as a dimensional-analysis technique.

Dimensional constants may vary from case to case but are held constant during a given run. In Eq. (5.5) they are S_0, V_0, and g, and in Eq. (5.6) they are ρ, g, and C. They all have dimensions and conceivably could be nondimensionalized, but they are normally used to help nondimensionalize the variables in the problem.

Pure constants have no dimensions and never did. They arise from mathematical manipulations. In both Eqs. (5.5) and (5.6) they are $\frac{1}{2}$ and the exponent 2, both of which came from an integration: $\int t\,dt = \frac{1}{2}t^2$, $\int V\,dV = \frac{1}{2}V^2$. Other common dimensionless constants are π and e.

Note that integration and differentiation of an equation may change the dimensions but not the homogeneity of the equation. For example, integrate or differentiate Eq. (5.5):

$$\int S\,dt = S_0t + \tfrac{1}{2}V_0t^2 + \tfrac{1}{6}gt^3 \tag{5.8a}$$

$$\frac{dS}{dt} = V_0 + gt \tag{5.8b}$$

In the integrated form (5.8*a*) every term has dimensions of $\{LT\}$, while in the derivative form (5.8*b*) every term is a velocity $\{LT^{-1}\}$.

Finally, there are some physical variables that are naturally dimensionless by virtue of their definition as ratios of dimensional quantities. Some examples are strain (change in length per unit length), Poisson's ratio (ratio of transverse strain to longitudinal strain), and specific gravity (ratio of density to standard water density). All angles are dimensionless (ratio of arc length to radius) and should be taken in radians for this reason.

The motive behind dimensional analysis is that any dimensionally homogeneous equation can be written in an entirely equivalent nondimensional form which is more compact. Usually there is more than one method of presenting one's dimensionless data or theory. Let us illustrate these concepts more thoroughly by using the falling-body relation (5.5) as an example.

Ambiguity: The Choice of Variables and Scaling Parameters[1]

Equation (5.5) is familiar and simple, yet illustrates most of the concepts of dimensional analysis. It contains five terms (S, S_0, V_0, t, g) which we may divide, in our thinking, into variables and parameters. The *variables* are the things which we wish to plot, the basic output of the experiment or theory: in this case, S versus t. The *parameters* are those quantities whose effect upon the variables we wish to know: in this case S_0, V_0, and g. Almost any engineering study can be subdivided in this manner.

To nondimensionalize our results, we need to know how many dimensions are contained among our variables and parameters: in this case, only two, length $\{L\}$ and time $\{T\}$. Check each term to verify this:

$$\{S\} = \{S_0\} = \{L\} \qquad \{t\} = \{T\} \qquad \{V_0\} = \{LT^{-1}\} \qquad \{g\} = \{LT^{-2}\}$$

Among our parameters, we therefore select two to be *scaling parameters,* used to define dimensionless variables. What remains will be the "basic" parameter(s) whose effect we wish to show in our plot. These choices will not affect the content of our data, only the form of their presentation. Clearly there is ambiguity in these choices, something that often vexes the beginning experimenter. But the ambiguity is deliberate. Its purpose is to show a particular effect, and the choice is yours to make.

For the falling-body problem, we select any two of the three parameters to be scaling parameters. Thus we have three options. Let us discuss and display them in turn.

Option 1: Scaling parameters S_0 and V_0: the effect of gravity g.

First use the scaling parameters (S_0, V_0) to define dimensionless (*) displacement and time. There is only one suitable definition for each:[2]

$$S^* = \frac{S}{S_0} \qquad t^* = \frac{V_0 t}{S_0} \tag{5.9}$$

Substitute these variables into Eq. (5.5) and clean everything up until each term is dimensionless. The result is our first option:

$$S^* = 1 + t^* + \frac{1}{2}\alpha t^{*2} \qquad \alpha = \frac{gS_0}{V_0^2} \tag{5.10}$$

This result is shown plotted in Fig. 5.1*a*. There is a single dimensionless parameter α, which shows here the effect of gravity. It cannot show the direct effects of S_0 and V_0, since these two are hidden in the ordinate and abscissa. We see that gravity increases the parabolic rate of fall for $t^* > 0$, but not the initial slope at $t^* = 0$. We would learn the same from falling-body data, and the plot, within experimental accuracy, would look like Fig. 5.1*a*.

[1] I am indebted to Prof. Jacques Lewalle of Syracuse University for suggesting, outlining, and clarifying this entire discussion.

[2] Make them *proportional* to S and t. Do not define dimensionless terms upside down: S_0/S or $S_0/(V_0 t)$. The plots will look funny, users of your data will be confused, and your supervisor will be angry. It is not a good idea.

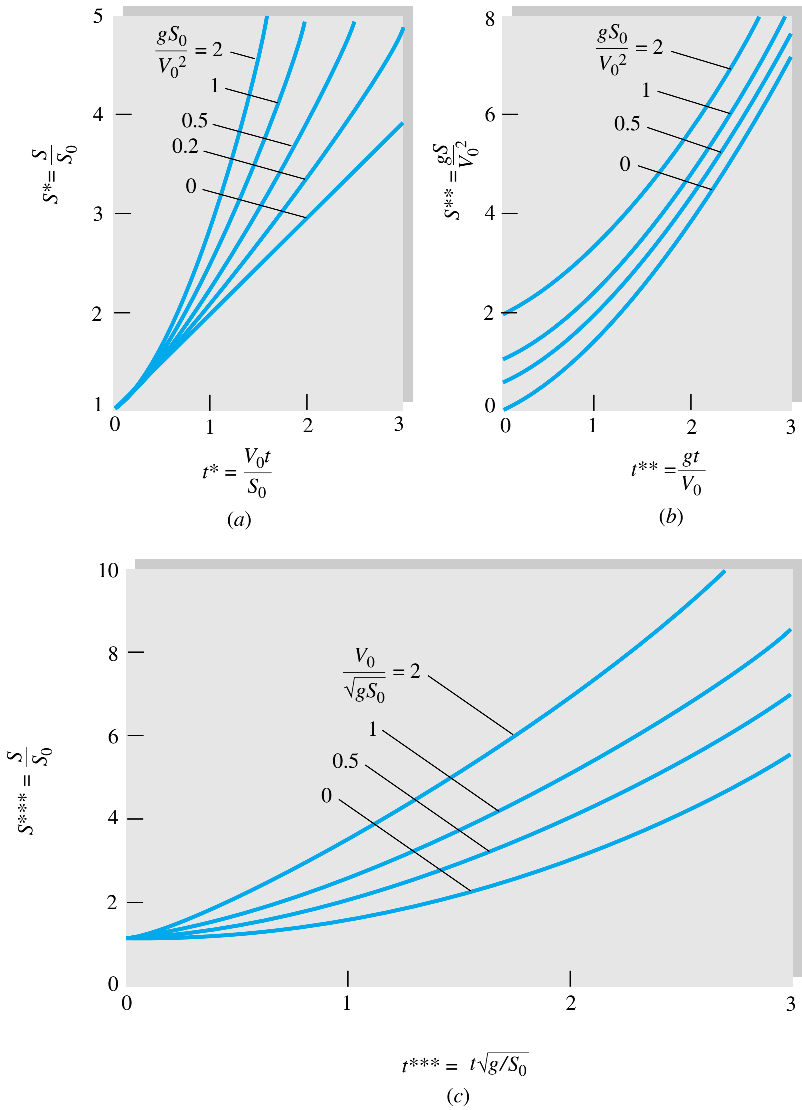

Fig. 5.1 Three entirely equivalent dimensionless presentations of the falling-body problem, Eq. (5.5): the effect of (*a*) gravity, (*b*) initial displacement, and (*c*) initial velocity. All plots contain the same information.

Option 2: Scaling parameters V_0 and g: the effect of initial displacement S_0.

Now use the new scaling parameters (V_0, g) to define dimensionless (**) displacement and time. Again there is only one suitable definition:

$$S^{**} = \frac{Sg}{V_0^2} \qquad t^{**} = t\frac{g}{V_0} \tag{5.11}$$

Substitute these variables into Eq. (5.5) and clean everything up again. The result is our second option:

$$S^{**} = \alpha + t^{**} + \frac{1}{2}t^{**2} \qquad \alpha = \frac{gS_0}{V_0^2} \tag{5.12}$$

This result is plotted in Fig. 5.1*b*. The same single parameter α again appears and here shows the effect of initial *displacement,* which merely moves the curves upward without changing their shape.

Option 3: Scaling parameters S_0 and g: the effect of initial speed V_0.

Finally use the scaling parameters (S_0, g) to define dimensionless (***) displacement and time. Again there is only one suitable definition:

$$S^{***} = \frac{S}{S_0} \qquad t^{***} = t\left(\frac{g}{S_0}\right)^{1/2} \tag{5.13}$$

Substitute these variables into Eq. (5.5) and clean everything up as usual. The result is our third and final option:

$$S^{***} = 1 + \beta t^{***} + \frac{1}{2}t^{***2} \qquad \beta = \frac{1}{\sqrt{\alpha}} = \frac{V_0}{\sqrt{gS_0}} \tag{5.14}$$

This final presentation is shown in Fig. 5.1*c*. Once again the parameter α appears, but we have redefined it upside down, $\beta = 1/\sqrt{\alpha}$, so that our display parameter V_0 is in the numerator and is linear. This is our free choice and simply improves the display. Figure 5.1*c* shows that initial *velocity* increases the falling displacement and that the increase is proportional to time.

Note that, in all three options, the same parameter α appears but has a different meaning: dimensionless gravity, initial displacement, and initial velocity. The graphs, which contain exactly the same information, change their appearance to reflect these differences.

Whereas the original problem, Eq. (5.5), involved five quantities, the dimensionless presentations involve only three, having the form

$$S' = \text{fcn}(t', \alpha) \qquad \alpha = \frac{gS_0}{V_0^2} \tag{5.15}$$

The reduction $5 - 3 = 2$ should equal the number of fundamental dimensions involved in the problem $\{L, T\}$. This idea led to the pi theorem (Sec. 5.3).

The choice of scaling variables is left to the user, and the resulting dimensionless parameters have differing interpretations. For example, in the dimensionless drag-force formulation, Eq. (5.2), it is now clear that the scaling parameters were ρ, V, and L, since they appear in both the drag coefficient and the Reynolds number. Equation (5.2) can thus be interpreted as the variation of dimensionless *force* with dimensionless *viscosity,* with the scaling-parameter effects mixed between C_F and Re and therefore not immediately evident.

Suppose that we wish to study drag force versus *velocity*. Then we would not use V as a scaling parameter. We would use (ρ, μ, L) instead, and the final dimensionless function would become

$$C_F' = \frac{\rho F}{\mu^2} = \text{fcn(Re)} \qquad \text{Re} = \frac{\rho V L}{\mu} \tag{5.16}$$

In plotting these data, we would not be able to discern the effect of ρ or μ, since they appear in both dimensionless groups. The grouping C_F' again would mean dimensionless force, and Re is now interpreted as either dimensionless velocity or size.[3] The plot would be quite different compared to Eq. (5.2), although it contains exactly the same information. The development of parameters such as C'_F and Re from the initial variables is the subject of the pi theorem (Sec. 5.3).

Some Peculiar Engineering Equations

The foundation of the dimensional-analysis method rests on two assumptions: (1) The proposed physical relation is dimensionally homogeneous, and (2) all the relevant variables have been included in the proposed relation.

If a relevant variable is missing, dimensional analysis will fail, giving either algebraic difficulties or, worse, yielding a dimensionless formulation which does not resolve the process. A typical case is Manning's open-channel formula, discussed in Example 1.4:

$$V = \frac{1.49}{n} R^{2/3} S^{1/2} \tag{1}$$

Since V is velocity, R is a radius, and n and S are dimensionless, the formula is not dimensionally homogeneous. This should be a warning that (1) the formula changes if the *units* of V and R change and (2) if valid, it represents a very special case. Equation (1) in Example 1.4 (see above) predates the dimensional-analysis technique and is valid only for water in rough channels at moderate velocities and large radii in BG units.

Such dimensionally inhomogeneous formulas abound in the hydraulics literature. Another example is the Hazen-Williams formula [30] for volume flow of water through a straight smooth pipe

$$Q = 61.9 D^{2.63} \left(\frac{dp}{dx}\right)^{0.54} \tag{5.17}$$

where D is diameter and dp/dx is the pressure gradient. Some of these formulas arise because numbers have been inserted for fluid properties and other physical data into perfectly legitimate homogeneous formulas. We shall not give the units of Eq. (5.17) to avoid encouraging its use.

On the other hand, some formulas are "constructs" which cannot be made dimensionally homogeneous. The "variables" they relate cannot be analyzed by the dimensional-analysis technique. Most of these formulas are raw empiricisms convenient to a small group of specialists. Here are three examples:

$$B = \frac{25{,}000}{100 - R} \tag{5.18}$$

$$S = \frac{140}{130 + \text{API}} \tag{5.19}$$

$$0.0147 D_E - \frac{3.74}{D_E} = 0.26 t_R - \frac{172}{t_R} \tag{5.20}$$

Equation (5.18) relates the Brinell hardness B of a metal to its Rockwell hardness R. Equation (5.19) relates the specific gravity S of an oil to its density in degrees API.

[3] We were lucky to achieve a size effect because in this case L, a scaling parameter, did not appear in the drag coefficient.

Equation (5.20) relates the viscosity of a liquid in D_E, or degrees Engler, to its viscosity t_R in Saybolt seconds. Such formulas have a certain usefulness when communicated between fellow specialists, but we cannot handle them here. Variables like Brinell hardness and Saybolt viscosity are not suited to an $MLT\Theta$ dimensional system.

5.3 The Pi Theorem

There are several methods of reducing a number of dimensional variables into a smaller number of dimensionless groups. The scheme given here was proposed in 1914 by Buckingham [29] and is now called the *Buckingham pi theorem*. The name *pi* comes from the mathematical notation Π, meaning a product of variables. The dimensionless groups found from the theorem are power products denoted by Π_1, Π_2, Π_3, etc. The method allows the pis to be found in sequential order without resorting to free exponents.

The first part of the pi theorem explains what reduction in variables to expect:

> If a physical process satisfies the PDH and involves n dimensional variables, it can be reduced to a relation between only k dimensionless variables or Π's. The reduction $j = n - k$ equals the maximum number of variables which do not form a pi among themselves and is always less than or equal to the number of dimensions describing the variables.

Take the specific case of force on an immersed body: Eq. (5.1) contains five variables F, L, U, ρ, and μ described by three dimensions $\{MLT\}$. Thus $n = 5$ and $j \leq 3$. Therefore it is a good guess that we can reduce the problem to k pis, with $k = n - j \geq 5 - 3 = 2$. And this is exactly what we obtained: two dimensionless variables $\Pi_1 = C_F$ and $\Pi_2 = \text{Re}$. On rare occasions it may take more pis than this minimum (see Example 5.5).

The second part of the theorem shows how to find the pis one at a time:

> Find the reduction j, then select j scaling variables which do not form a pi among themselves.[4] Each desired pi group will be a power product of these j variables plus one additional variable which is assigned any convenient nonzero exponent. Each pi group thus found is independent.

To be specific, suppose that the process involves five variables

$$v_1 = f(v_2, v_3, v_4, v_5)$$

Suppose that there are three dimensions $\{MLT\}$ and we search around and find that indeed $j = 3$. Then $k = 5 - 3 = 2$ and we expect, from the theorem, two and only two pi groups. Pick out three convenient variables which do *not* form a pi, and suppose these turn out to be v_2, v_3, and v_4. Then the two pi groups are formed by power products of these three plus one additional variable, either v_1 or v_5:

$$\Pi_1 = (v_2)^a(v_3)^b(v_4)^c v_1 = M^0L^0T^0 \qquad \Pi_2 = (v_2)^a(v_3)^b(v_4)^c v_5 = M^0L^0T^0$$

Here we have arbitrarily chosen v_1 and v_5, the added variables, to have unit exponents. Equating exponents of the various dimensions is guaranteed by the theorem to give unique values of a, b, and c for each pi. And they are independent because only Π_1

[4] Make a clever choice here because all pis will contain these j variables in various groupings.

Table 5.1 Dimensions of Fluid-Mechanics Properties

Quantity	Symbol	Dimensions $MLT\Theta$	Dimensions $FLT\Theta$
Length	L	L	L
Area	A	L^2	L^2
Volume	$\mathcal{V}$	L^3	L^3
Velocity	V	LT^{-1}	LT^{-1}
Acceleration	dV/dt	LT^{-2}	LT^{-2}
Speed of sound	a	LT^{-1}	LT^{-1}
Volume flow	Q	L^3T^{-1}	L^3T^{-1}
Mass flow	$\dot{m}$	MT^{-1}	FTL^{-1}
Pressure, stress	p, σ	$ML^{-1}T^{-2}$	FL^{-2}
Strain rate	$\dot{\epsilon}$	T^{-1}	T^{-1}
Angle	θ	None	None
Angular velocity	ω	T^{-1}	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}
Surface tension	Υ	MT^{-2}	FL^{-1}
Force	F	MLT^{-2}	F
Moment, torque	M	ML^2T^{-2}	FL
Power	P	ML^2T^{-3}	FLT^{-1}
Work, energy	W, E	ML^2T^{-2}	FL
Density	ρ	ML^{-3}	FT^2L^{-4}
Temperature	T	Θ	Θ
Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Specific weight	γ	$ML^{-2}T^{-2}$	FL^{-3}
Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$	$FT^{-1}\Theta^{-1}$
Expansion coefficient	β	Θ^{-1}	Θ^{-1}

contains υ_1 and only Π_2 contains υ_5. It is a very neat system once you get used to the procedure. We shall illustrate it with several examples.

Typically, six steps are involved:

1. List and count the n variables involved in the problem. If any important variables are missing, dimensional analysis will fail.
2. List the dimensions of each variable according to {MLTΘ} or {FLTΘ}. A list is given in Table 5.1.
3. Find j. Initially guess j equal to the number of different dimensions present, and look for j variables which do not form a pi product. If no luck, reduce j by 1 and look again. With practice, you will find j rapidly.
4. Select j scaling parameters which do not form a pi product. Make sure they please you and have some generality if possible, because they will then appear in every one of your pi groups. Pick density or velocity or length. Do not pick surface tension, e.g., or you will form six different independent Weber-number parameters and thoroughly annoy your colleagues.
5. Add one additional variable to your j repeating variables, and form a power product. Algebraically find the exponents which make the product dimensionless. Try to arrange for your output or *dependent* variables (force, pressure drop, torque, power) to appear in the numerator, and your plots will look better. Do

this sequentially, adding one new variable each time, and you will find all $n - j = k$ desired pi products.

6. Write the final dimensionless function, and check your work to make sure all pi groups are dimensionless.

EXAMPLE 5.2

Repeat the development of Eq. (5.2) from Eq. (5.1), using the pi theorem.

Solution

Step 1 Write the function and count variables:

$$F = f(L, U, \rho, \mu) \quad \text{there are five variables } (n = 5)$$

Step 2 List dimensions of each variable. From Table 5.1

F	L	U	ρ	μ
$\{MLT^{-2}\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$

Step 3 Find j. No variable contains the dimension Θ, and so j is less than or equal to 3 (MLT). We inspect the list and see that L, U, and ρ cannot form a pi group because only ρ contains mass and only U contains time. Therefore j does equal 3, and $n - j = 5 - 3 = 2 = k$. The pi theorem guarantees for this problem that there will be exactly two independent dimensionless groups.

Step 4 Select repeating j variables. The group L, U, ρ we found in step 3 will do fine.

Step 5 Combine L, U, ρ with one additional variable, in sequence, to find the two pi products.

First add force to find Π_1. You may select *any* exponent on this additional term as you please, to place it in the numerator or denominator to any power. Since F is the output, or dependent, variable, we select it to appear to the first power in the numerator:

$$\Pi_1 = L^a U^b \rho^c F = (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

Equate exponents:

$$\begin{aligned} &\text{Length:} & a + b - 3c + 1 &= 0 \\ &\text{Mass:} & c + 1 &= 0 \\ &\text{Time:} & -b \qquad -2 &= 0 \end{aligned}$$

We can solve explicitly for

$$a = -2 \qquad b = -2 \qquad c = -1$$

Therefore

$$\Pi_1 = L^{-2} U^{-2} \rho^{-1} F = \frac{F}{\rho U^2 L^2} = C_F \qquad \textit{Ans.}$$

This is exactly the right pi group as in Eq. (5.2). By varying the exponent on F, we could have found other equivalent groups such as $UL\rho^{1/2}/F^{1/2}$.

Finally, add viscosity to L, U, and ρ to find Π_2. Select any power you like for viscosity. By hindsight and custom, we select the power -1 to place it in the denominator:

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^{-1} = M^0 L^0 T^0$$

Equate exponents:

$$\begin{aligned} &\text{Length:} && a + b - 3c + 1 = 0 \\ &\text{Mass:} && c - 1 = 0 \\ &\text{Time:} && -b \quad\quad + 1 = 0 \end{aligned}$$

from which we find

$$a = b = c = 1$$

Therefore $$\Pi_2 = L^1 U^1 \rho^1 \mu^{-1} = \frac{\rho U L}{\mu} = \text{Re}$$ *Ans.*

We know we are finished; this is the second and last pi group. The theorem guarantees that the functional relationship must be of the equivalent form

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho U L}{\mu}\right)$$ *Ans.*

which is exactly Eq. (5.2).

EXAMPLE 5.3

Reduce the falling-body relationship, Eq. (5.5), to a function of dimensionless variables. Why are there three different formulations?

Solution

Write the function and count variables

$$S = f(t, S_0, V_0, g) \quad \text{five variables } (n = 5)$$

List the dimensions of each variable, from Table 5.1:

S	t	S_0	V_0	g
$\{L\}$	$\{T\}$	$\{L\}$	$LT^{-1}\}$	$\{LT^{-2}\}$

There are only two primary dimensions (L, T), so that $j \le 2$. By inspection we can easily find two variables which cannot be combined to form a pi, for example, V_0 and g. Then $j = 2$, and we expect $5 - 2 = 3$ pi products. Select j variables among the parameters S_0, V_0, and g. Avoid S and t since they are the dependent variables, which should not be repeated in pi groups.

There are three different options for repeating variables among the group (S_0, V_0, g). Therefore we can obtain three different dimensionless formulations, just as we did informally with the falling-body equation in Sec. 5.2. Take each option in turn:

1. Choose S_0 and V_0 as repeating variables. Combine them in turn with (S, t, g):

$$\Pi_1 = S^1 S_0^a V_0^b \quad \Pi_2 = t^1 S_0^c V_0^d \quad \Pi_3 = g^1 S_0^e V_0^f$$

Set each power product equal to $L^0 T^0$, and solve for the exponents (a, b, c, d, e, f). Please allow us to give the results here, and you may check the algebra as an exercise:

$$a = -1 \quad b = 0 \quad c = -1 \quad d = 1 \quad e = 1 \quad f = -2$$

$$\Pi_1 = S^* = \frac{S}{S_0} \qquad \Pi_2 = t^* = \frac{V_0 t}{S_0} \qquad \Pi_3 = \alpha = \frac{g S_0}{V_0^2} \qquad \textit{Ans.}$$

Thus, for option 1, we know that $S^* = \text{fcn}(t^*, \alpha)$. We have found, by dimensional analysis, the same variables as in Eq. (5.10). But here there is no *formula* for the functional relation — we might have to experiment with falling bodies to establish Fig. 5.1*a*.

2. Choose V_0 and g as repeating variables. Combine them in turn with (S, t, S_0):

$$\Pi_1 = S^1 V_0^a g^b \quad \Pi_2 = t^1 V_0^c g^d \quad \Pi_3 = S_0^1 V_0^e g^f$$

Set each power product equal to $L^0 T^0$, and solve for the exponents (a, b, c, d, e, f). Once more allow us to give the results here, and you may check the algebra as an exercise.

$$a = -2 \quad b = 1 \quad c = -1 \quad d = 1 \quad e = 1 \quad f = -2$$

$$\Pi_1 = S^{**} = \frac{Sg}{V_0^2} \quad \Pi_2 = t^{**} = \frac{tg}{V_0} \quad \Pi_3 = \alpha = \frac{g S_0}{V_0^2} \qquad \textit{Ans.}$$

Thus, for option 2, we now know that $S^{**} = \text{fcn}(t^{**}, \alpha)$. We have found, by dimensional analysis, the same groups as in Eq. (5.12). The data would plot as in Fig. 5.1*b*.

3. Finally choose S_0 and g as repeating variables. Combine them in turn with (S, t, V_0):

$$\Pi_1 = S^1 S_0^a g^b \quad \Pi_2 = t^1 S_0^c g^d \quad \Pi_3 = V_0^1 S_0^e g^f$$

Set each power product equal to $L^0 T^0$, and solve for the exponents (a, b, c, d, e, f). One more time allow us to give the results here, and you may check the algebra as an exercise:

$$a = -1 \quad b = 0 \quad c = -\frac{1}{2} \quad d = \frac{1}{2} \quad e = -\frac{1}{2} \quad f = -\frac{1}{2}$$

$$\Pi_1 = S^{***} = \frac{S}{S_0} \quad \Pi_2 = t^{***} = t\sqrt{\frac{g}{S_0}} \quad \Pi_3 = \beta = \frac{V_0}{\sqrt{g S_0}} \qquad \textit{Ans.}$$

Thus, for option 3, we now know that $S^{***} = \text{fcn}(t^{***}, \beta = 1/\sqrt{\alpha})$. We have found, by dimensional analysis, the same groups as in Eq. (5.14). The data would plot as in Fig. 5.1*c*.

Dimensional analysis here has yielded the same pi groups as the use of scaling parameters with Eq. (5.5). Three different formulations appeared, because we could choose three different pairs of repeating variables to complete the pi theorem.

EXAMPLE 5.4

At low velocities (laminar flow), the volume flow Q through a small-bore tube is a function only of the tube radius R, the fluid viscosity μ, and the pressure drop per unit tube length dp/dx. Using the pi theorem, find an appropriate dimensionless relationship.

Solution

Write the given relation and count variables:

$$Q = f\left(R, \mu, \frac{dp}{dx}\right) \quad \text{four variables } (n = 4)$$

Make a list of the dimensions of these variables from Table 5.1:

Q	R	μ	dp/dx
$\{L^3T^{-1}\}$	$\{L\}$	$\{ML^{-1}T^{-1}\}$	$\{ML^{-2}T^{-2}\}$

There are three primary dimensions (M, L, T), hence $j \le 3$. By trial and error we determine that R, μ, and dp/dx cannot be combined into a pi group. Then $j = 3$, and $n - j = 4 - 3 = 1$. There is only *one* pi group, which we find by combining Q in a power product with the other three:

$$\Pi_1 = R^a \mu^b \left(\frac{dp}{dx}\right)^c Q^1 = (L)^a (ML^{-1}T^{-1})^b (ML^{-2}T^{-2})^c (L^3T^{-1})$$
$$= M^0 L^0 T^0$$

Equate exponents:

Mass: $\qquad b + c = 0$

Length: $\qquad a - b - 2c + 3 = 0$

Time: $\qquad -b - 2c - 1 = 0$

Solving simultaneously, we obtain $a = -4$, $b = 1$, $c = -1$. Then

$$\Pi_1 = R^{-4}\mu^1 \left(\frac{dp}{dx}\right)^{-1} Q$$

or

$$\Pi_1 = \frac{Q\mu}{R^4(dp/dx)} = \text{const} \qquad \textit{Ans.}$$

Since there is only one pi group, it must equal a dimensionless constant. This is as far as dimensional analysis can take us. The laminar-flow theory of Sec. 6.4 shows that the value of the constant is $\pi/8$.

EXAMPLE 5.5

Assume that the tip deflection δ of a cantilever beam is a function of the tip load P, beam length L, area moment of inertia I, and material modulus of elasticity E; that is, $\delta = f(P, L, I, E)$. Rewrite this function in dimensionless form, and comment on its complexity and the peculiar value of j.

Solution

List the variables and their dimensions:

δ	P	L	I	E
$\{L\}$	$\{MLT^{-2}\}$	$\{L\}$	$\{L^4\}$	$\{ML^{-1}T^{-2}\}$

There are five variables ($n = 5$) and three primary dimensions (M, L, T), hence $j \le 3$. But try as we may, we *cannot* find any combination of three variables which does not form a pi group. This is because $\{M\}$ and $\{T\}$ occur only in P and E and only in the same form, $\{MT^{-2}\}$. Thus we have encountered a special case of $j = 2$, which is less than the number of dimensions (M, L, T). To gain more insight into this peculiarity, you should rework the problem, using the (F, L, T) system of dimensions.

With $j = 2$, we select L and E as two variables which cannot form a pi group and then add other variables to form the three desired pis:

$$\Pi_1 = L^a E^b I^1 = (L)^a (ML^{-1}T^{-2})^b (L^4) = M^0 L^0 T^0$$

from which, after equating exponents, we find that $a = -4$, $b = 0$, or $\Pi_1 = I/L^4$. Then

$$\Pi_2 = L^a E^b P^1 = (L)^a (ML^{-1}T^{-2})^b (MLT^{-2}) = M^0 L^0 T^0$$

from which we find $a = -2$, $b = -1$, or $\Pi_2 = P/(EL^2)$, and

$$\Pi_3 = L^a E^b \delta^1 = (L)^a (ML^{-1}T^{-2})^b (L) = M^0 L^0 T^0$$

from which $a = -1$, $b = 0$, or $\Pi_3 = \delta/L$. The proper dimensionless function is $\Pi_3 = f(\Pi_2, \Pi_1)$, or

$$\frac{\delta}{L} = f\left(\frac{P}{EL^2}, \frac{I}{L^4}\right) \qquad \textit{Ans.} \quad (1)$$

This is a complex three-variable function, but dimensional analysis alone can take us no further.

We can "improve" Eq. (1) by taking advantage of some physical reasoning, as Langhaar points out [8, p. 91]. For small elastic deflections, δ is proportional to load P and inversely proportional to moment of inertia I. Since P and I occur separately in Eq. (1), this means that Π_3 must be proportional to Π_2 and inversely proportional to Π_1. Thus, for these conditions,

$$\frac{\delta}{L} = (\text{const}) \frac{P}{EL^2} \frac{L^4}{I}$$

or

$$\delta = (\text{const}) \frac{PL^3}{EI} \qquad (2)$$

This could not be predicted by a pure dimensional analysis. Strength-of-materials theory predicts that the value of the constant is $\frac{1}{3}$.

5.4 Nondimensionalization of the Basic Equations

We could use the pi-theorem method of the previous section to analyze problem after problem after problem, finding the dimensionless parameters which govern in each case. Textbooks on dimensional analysis [for example, 7] do this. An alternate and very powerful technique is to attack the basic equations of flow from Chap. 4. Even though these equations cannot be solved in general, they will reveal basic dimensionless parameters, e.g., Reynolds number, in their proper form and proper position, giving clues to when they are negligible. The boundary conditions must also be nondimensionalized.

Let us briefly apply this technique to the incompressible-flow continuity and momentum equations with constant viscosity:

Continuity:

$$\nabla \cdot \mathbf{V} = 0 \qquad (5.21a)$$